Meson Effects in *n-p* Capture*

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A contribution to the *n-p* capture amplitude which results from the modification of the nucleon current operators by the nuclear interaction is calculated. The two-nucleon states are represented by Heitler-London states and the capture amplitude is related to single-nucleon matrix elements by means of an expansion corresponding to the exchange of mesons between the nucleons. These single-nucleon matrix elements are evaluated using the fixed-source theory. The one-meson excited Heitler-London state gives a contribution similar to the isobar model of Austern; the role of this state in the nuclear potential is also discussed. If a deuteron wave function with a relatively large D-state probability is used, the interaction effect is about 3% of the phenomenological cross section.

I. INTRODUCTION

THE thermal *n-p* capture process provides a simple test of theories of the nuclear force at very low energy. The transition amplitude depends on the initial- and final-state wave functions at small distances of separation as well as on the asymptotic form of the wave functions. However, it is also necessary to know the nature of the radiative interaction, and this means knowledge of the electromagnetic currents generated by the motion of the nucleons. These currents depend strongly on the internal structure of the nucleons, and when two nucleons are close together there is a modification of this internal structure by the interaction between the nucleons. The total electromagnetic current can be expressed as a "classical" current, where each nucleon is assumed to have the same structure it would have if free, and corrections which are called interaction currents.

At thermal energies, the capture transition is magnetic dipole and proceeds only from an incident S wave; the other quantum numbers of the incident state are isospin one and spin zero. A recent calculation by Austern and Rost,[1] using an effective range expansion, has shown that the classical current alone gives a theoretical cross section $(9.5\pm3.5)\%$ lower than the experimental value. They suggested that the discrepancy is due to interaction currents. There is another reason to consider that interaction currents may play a role. Meson theories of the tensor potential indicate that P_D, the D-state probability of the deuteron, may be considerably larger than 4%, the value required to fit the deuteron magnetic moment without interaction effect. Detailed studies of the medium energy photodisintegration of the deuteron[2] indicates that P_D may be nearly 7%. In view of the large interaction effect in the deuteron magnetic moment, it would not be surprising to find interaction currents playing a significant role in the capture process.

Some attempts,[3] based on a Tamm-Dancoff approximation have been made to calculate the interaction effect in both the capture process and the deuteron magnetic moment, but these attempts gave negligible interaction effect. However, this result may be partially discounted because of the difficulties inherent to the Tamm-Dancoff method, and because the effect of the resonant pi-nucleon scattering state was not included. Recently, the Heitler-London method[4] has been applied to the deuteron magnetic moment problem,[5] the calculated interaction effects were large enough to be consistent with $P_D\sim6$ or 7%. The present work attempts to apply the same techniques to the capture amplitude.

In the approximations used here, the Heitler-London method relates the capture amplitude to the pi-nucleon vertex function and the pi-nucleon scattering matrix, and the contribution of the resonant pi-nucleon scattering state is included in a straightforward manner. This contribution to the interaction current is the dominant effect which may be pictured as follows. An exchanged meson excites one nucleon into the resonant scattering state, the nucleons propagate for a short time, and then the scattering meson is absorbed and the photon emitted via the magnetic-dipole photoproduction process. The situation is similar to that of Austern's isobar model[6] used to explain the peak in the deuteron photodisintegration cross section.

Required to evaluate this contribution are the pi-nucleon vertex function and the transition matrix in the $\frac{3}{2}$-$\frac{3}{2}$ state. Since the static-source theory successfully describes these two properties of the pi-nucleon system, that theory will be used to provide the off-energy-shell extrapolations needed in this calculation.

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[1] N. Austern and E. Rost, Phys. Rev. **117**, 1506 (1960).
[2] J. J. de Swart and R. E. Marshak, Physica **25**, 1001 (1959); Phys. Rev. **111**, 272 (1958); A. F. Nicholson and G. E. Brown, Proc. Phys. Soc. (London) **73**, 221 (1959); M. L. Rustgi, W. Zernik, G. Breit, and D. J. Andrews, Phys. Rev. **120**, 1881 (1960).
[3] M. Sugawara, Progr. Theoret. Phys. (Kyoto) **14**, 535 (1955); Phys. Rev. **99**, 1601 (1955).
[4] R. E. Cutkosky, Phys. Rev. **112**, 1027 (1958).
[5] H. D. Young and R. E. Cutkosky, Phys. Rev. **117**, 595 (1960).
[6] N. Austern, Phys. Rev. **100**, 1522 (1955).

II. THE MODEL

We will follow the approach introduced by Cutkosky[4] and further described by Pendleton[7] wherein the two nucleon states are expanded in terms of Heitler-London states. These states are perhaps best described by their explicit construction. We first introduce the operators, which acting on the physical vacuum state, create physical nucleon states, including scattered mesons where appropriate. The states produced are normalized eigenstates of both the total Hamiltonian H and the momentum operator, thus

$$F^\dagger(p)|0\rangle=|p\rangle, H|p\rangle=E_p|p\rangle,$$
$$F_k^\dagger(p)|0\rangle=|p,k\rangle, H|p,k\rangle=(E_p+\omega_k)|p,k\rangle,$$

where p is the nucleon variable including spin and isotopic spin, and k is the meson variable including isotopic spin, and[8]

$$E_p^2=p^2+m^2, \quad \omega_k^2=k^2+1$$

states with more scattering mesons are possible, but will not be necessary here. Of the many possible $F^\dagger$ that fulfill the above equations, we choose the one that contains only meson creation operators $a_k^\dagger$.[9] The basic Heitler-London state is defined by

$$|\Phi(p_1p_2)\rangle=\tfrac{1}{2}F^\dagger(p_1)F^\dagger(p_2)|0\rangle, \tag{1}$$

and the first excited Heitler-London state by

$$|\Phi_k(p_1p_2)\rangle=\tfrac{1}{2}(F_k^\dagger(p_1)F^\dagger(p_2) +F^\dagger(p_1)F_k^\dagger(p_2)-a_k^\dagger F^\dagger(p_1)F^\dagger(p_2))|0\rangle. \tag{2}$$

Heitler-London states with more mesons can be easily constructed by obvious generalization. With the assumption that these form a normable and complete set of states, any two nucleon eigenstate may be represented by the expansion

$$|\psi\rangle=\sum_{p_1p_2}\varphi(p_1p_2)|\Phi(p_1p_2)\rangle +\sum_{kp_1p_2}\varphi_k(p_1p_2)|\Phi_k(p_1p_2)\rangle+\cdots. \tag{3}$$

In the present work, the expansion will not be carried any further than the first two terms.

For bound states, coupled equations for the amplitudes φ, φ_k can be obtained from the variational principle

$$\delta\langle\psi|H-E|\psi\rangle=0 \tag{4}$$

by varying φ and φ_k. For scattering states similar equations can be obtained by the method of Pendleton.[7] These equations will involve various matrix elements of the Heitler-London states, and a notation for these matrix elements will be introduced first. The normalization of the basic state is

$$\langle\Phi(p_1p_2)|\Phi(p_1'p_2')\rangle=1/2(2\pi)^6 \times[\delta^3(p_2-p_2')\delta^3(p_1-p_1')-\delta^3(p_2-p_1')\delta^3(p_1-p_2')] +A(p_1p_2;p_1'p_2'). \tag{5}$$

The function A describes the change in normalization of the basic state due to overlap between the two nucleons. It will be convenient to occasionally use a matrix notation where the nucleon (but not the meson) variables are suppressed, and the antisymmetric occurrence of the delta functions is understood; thus

$$\langle\Phi|\Phi\rangle=1+A\equiv\Gamma^{-2}. \tag{6}$$

In a similar manner, the other matrix elements needed are written

$$\langle\Phi|H|\Phi\rangle=K\Gamma^{-2}+U, \tag{7}$$

where K is the kinetic energy of the nucleons,

$$\langle\Phi_k|\Phi_{k'}\rangle=\delta_{kk'}\Gamma^{-2}+A_{kk'}, \tag{8}$$

$$\langle\Phi_k|H|\Phi_{k'}\rangle=[K+1/2(\omega_k+\omega_{k'})] \times[\delta_{kk'}\Gamma^{-2}+A_{kk'}]+\delta_{kk'}U+U_{kk'}, \tag{9}$$

and

$$\langle\Phi_k|H-E|\Phi\rangle=W_k. \tag{10}$$

The usefulness of the Heitler-London states is that the functions A, U, W_k, etc., can be expanded in a series that corresponds to increasing numbers of mesons being exchanged between the nucleons.[4,7] Each term of these series can be related to observable (in general, off-energy-shell) properties of single nucleon states. The coupled equations for the amplitudes are

$$(K-E)\Gamma^{-2}\varphi+U\varphi+\Sigma_k W_k^\dagger\varphi_k=0, \tag{11}$$

$$\Sigma_l[(K+\omega_l-E)(\delta_{kl}\Gamma^{-2}+A_{kl})+\delta_{kl}U+U_{kl}]\varphi_l=-W_k\varphi. \tag{12}$$

For bound states, the amplitude φ must vanish for large spatial separations, for scattering states φ must satisfy outgoing or incoming wave boundary conditions for large spatial separations.

Cutkosky's[10] extensive calculation of nuclear potentials using Heitler-London states provides valuable insight to the solution of these equations. He found that the basic Heitler-London state alone provides a good approximation to the deuteron. That is, φ_k is small and φ is consistent with current suitably normalized phenomenological deuteron wave functions. For isospin one states, such as the incident $^1S_0$ state of this calculation, his conclusion was that the excited Heitler-London plays an important role. It was suggested that an adequate description of these states could only be reached by a solution of the equations for both amplitudes φ and φ_k. While the exact solution of the coupled equations is not possible, a reasonable approximation

[7] H. N. Pendleton, Phys. Rev. **131**, 1833 (1963).
[8] We use units such that $\hbar$, c and the mass of the pion are one.
[9] We mean here physical meson creation operators. We assume that they may be derived by applying a unitary transformation to the bare meson operators so that the dressed operators obey canonical commutation relations.
[10] R. E. Cutkosky, Phys. Rev. **116**, 1272 (1959).

is not difficult. Let us assume that an exact solution would correctly describe the $^1S_0$ state. Then φ would closely resemble the phenomenological wave functions for this state. Our approximation is to assume for φ a suitably normalized phenomenological wave function, then a formal solution for the excited-state amplitude is given by

$$\varphi_k = -[\Sigma_l(K+\omega_l-E)(\delta_{kl}\Gamma^{-2}+A_{kl}) +\delta_{kl}U+U_{kl}]^{-1}W_k\varphi = -GW_k\varphi. \quad (13)$$

If this procedure were correct, then the solution of Eq. (11) including the recoupling of the excited-state amplitude, should reproduce the φ originally chosen. With the recoupling, Eq. (11) becomes

$$(K-E+U-\Sigma_k W_k^\dagger G W_k)\varphi = 0. \quad (14)$$

An equivalent criterion of the validity of our procedure is that the potential

$$U-\Sigma_k W_k^\dagger G W_k$$

be consistent with current phenomenological two-nucleon potentials. While this potential is briefly discussed later, the main purpose of this calculation is not to calculate nuclear potentials, but to obtain a reasonable approximation to the excited-state amplitude φ_k. We will take φ to be a phenomenological wave function, and the excitation and propagation of the excited state will be treated in an approximation that corresponds to one meson being exchanged between the nucleons. This means that only the first term in the expansion of W_k is kept and the functions U_{kl} and A_{kl} are dropped. The neglect of these functions correspond to calculating effects of the excited meson only at its emission and absorption. The net results of these approximations are that we take, for the deuteron amplitude

$$\varphi_D = \Gamma\psi_D, \quad (15)$$

where ψ_D is a phenomenological wave function for the deuteron, normalized to unity. And for the $^1S_0$ final state we take

$$\varphi = \Gamma\psi,$$
$$\varphi_k = -[(K+\omega_k-E)\Gamma^{-2}+U]^{-1}W_k\psi, \quad (16)$$

where ψ is a phenomenological wave function that fits the n-p scattering data. The functions W_k and Γ are calculated only to first order in exchanged mesons.

III. THE TRANSITION AMPLITUDE

To first order in the electromagnetic interaction, and with the present approximation for the two-nucleon states, the capture transition amplitude is

$$\psi_D^\dagger\Gamma\langle\Phi|J_\gamma|\Phi\rangle\Gamma\psi + \Sigma_k\psi_D^\dagger\Gamma\langle\Phi|J_\gamma|\Phi_k\rangle\varphi_k. \quad (17)$$

The second term is the contribution of the excited Heitler-London state, and will be treated in the next section. We continue here with the first term. In the normalization factor, $\Gamma=(1+A)^{-1/2}$, A contains the one-meson-exchange corrections and is presumed small compared to one; we will use the approximation $\Gamma\sim 1-\frac{1}{2}A$. Further, the $-\frac{1}{2}A$ will be kept only when it multiplies terms that involve no meson exchanges. Finally, the currents of exchanged mesons will not be calculated. Sugawara[3] has calculated these contributions, and they are both small and incoherent with the classical current. The contribution of the basic Heitler-London state, including terms up to one meson exchange, is

$$\tfrac{1}{2}\psi_D^\dagger(1-\tfrac{1}{2}A)\langle p_2'|p_2\rangle\langle p_1'|J_\gamma|p_1\rangle_{as}(1-\tfrac{1}{2}A)\psi + \tfrac{1}{2}\Sigma_l\psi_D^\dagger\langle p_2'|a_l|p_2\rangle\langle p_1'|a_l^\dagger J_\gamma|p_1\rangle_{as} + \langle p_2'|a_l^\dagger|p_2\rangle\langle p_1'|J_\gamma a_l|p_1\rangle_{as}\psi. \quad (18)$$

A mixed notation is being used whereby the nucleon variables are explicitly displayed in some of the terms. The subscript as means to add the terms with p_1, p_2 interchanged, and with p_1', p_2' interchanged. In the last term, it is convenient to make a closure expansion between the operators $a_l^\dagger J_\gamma$ and $J_\gamma a_l$; however, only the single-nucleon and the pi-nucleon scattering states will be kept in the closure expansion. When due account of the antisymmetry in the nucleon variables is taken, the first term of Eq. (18) is of the form

$$([1-A]\psi_D, J_\gamma[1-A]\psi),$$

and the terms with the single-nucleon state in the closure expansion is of the form

$$(\psi_D, [J_\gamma A - AJ_\gamma]\psi).$$

The sum of these, neglecting A^2 terms, is

$$(\psi_D, J_\gamma\psi), \quad (19)$$

which is the classical amplitude discussed by Austern and Rost.[1] Using the nonrelativistic limit of the nucleon electromagnetic vertex this is

$$\int d^3x\psi_D(x)\exp(-i\boldsymbol{\gamma}\cdot\tfrac{1}{2}\mathbf{x})i\boldsymbol{\tau}_3(\mu_p-\mu_n)\boldsymbol{\sigma}\cdot\boldsymbol{\gamma} \times\boldsymbol{\varepsilon}\psi(x)/(8\gamma)^{1/2}, \quad (20)$$

where $\boldsymbol{\tau}_3$ is the isotopic spin operator for a nucleon, $\boldsymbol{\sigma}$ the spin operator, $\boldsymbol{\gamma}$ and $\boldsymbol{\varepsilon}$ are, respectively, the photon momentum and polarization vectors, and γ is the photon energy.[8] In all that follows the phase of the photon, $\exp(i\boldsymbol{\gamma}\cdot\frac{1}{2}\mathbf{x})$, will be taken to be 1. The remaining contribution of the basic Heitler-London state is that from the pi-nucleon scattering state in the closure expansion

$$\tfrac{1}{2}\sum_{l,gm}\psi_D^\dagger\langle p_2'|a_l|p_2\rangle\langle p_1'|a_l^\dagger|gm\rangle\langle gm|J_\gamma|p_1\rangle_{as} + \langle p_2'|a_l^\dagger|p_2\rangle\langle p_1'|J_\gamma|gm\rangle\langle gm|a_l|p_1\rangle_{as}\psi. \quad (21)$$

The subscript as means to add the terms with p_1, p_2 and p_1', p_2' interchanged, but not to add any inter-

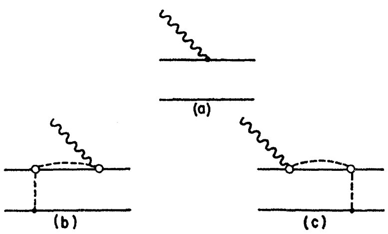

FIG. 1. Contributions of the basic Heitler-London state (a) The classical amplitude. (b) and (c) The one-meson-exchange, scattering intermediate state contributions.

changes involving the nucleon variable g. The contribution of the classical current and these terms are illustrated diagramatically in Fig. 1, the dots represent vertex functions, the circles represent T-matrix elements.

This part of the amplitude is now expressed in terms of matrix elements of the meson field operators and the photon current operator taken between physical nucleon states. These matrix elements are related to physical quantities by use of the identity

$$a_l|p\rangle=(E_p-H-\omega_l)^{-1}V_l|p\rangle\,,$$

where $V_l\equiv[H_{\text{int}},a_l]$.

The no-nucleon recoil approximation will always be used whereby the kinetic energies of the nucleon are dropped in the energy denominators. These matrix elements are now easily recognized from the theory of scattering; $\langle p_2|V_l|p_2'\rangle$ is the meson-nucleon renormalized vertex form factor, $\langle p_1|V_l|gm\rangle$ is the complex conjugate of the meson-nucleon elastic T matrix element, and $\langle gm|J_\gamma|p_1'\rangle$ is the photoproduction T matrix element. Since we are already restricted to nonrecoiling nucleons, these matrix elements can reasonably be taken directly from the Chew-Low-Wick fixed-source theory.[11] There is a difficulty, though, in that the matrix elements needed here are not in the Lorentz frame where the nucleons are at rest. Our matrix elements, being in the essentially noncovariant Schrödinger picture, have a complicated Lorentz transformation related to that of the Low equation. We shall assume that the matrix elements are the same in all frames, and consider the error involved as part of the no-nucleon-recoil approximation.

In the static theory the vertex form factor is given by

$$\langle p_2|V_l|p_2'\rangle=(2\pi)^3\delta^3(p_2+l-p_2')V(l)\,,$$

where $V(l)=i(2\omega_l)^{-1/2}\boldsymbol{\tau}_l\boldsymbol{\sigma}\cdot\mathbf{l}fv(l)$, f is the renormalized coupling constant and $v(l)$ is the cutoff function. The T matrix is given as

$$\langle p_1|V_l|gm\rangle=(2\pi)^3\delta^3(p_1+l-g-m)T_l^*(m)\,,$$

where

$$T_l(m)=\frac{-2\pi v(l)v(m)}{(\omega_l\omega_m)^{1/2}}\sum_\alpha\frac{P_\alpha(m,l)\exp(i\delta_\alpha(m))\sin\delta_\alpha(m)}{m^3v^2(m)}\,,$$

and where the P_α are the projection operators of Refs. 11. The magnetic dipole, isotopic vector part of the photoproduction matrix element is given approximately by[11]

$$\langle gm|J_\gamma|p_1\rangle=(\mu_p-\mu_n)(2fv(l))^{-1}(\omega_\gamma/\gamma)^{1/2}T_{\gamma\times\epsilon}(m)\,,\quad(22)$$

where the momentum conservation δ function is understood.

A rough description of the term corresponding to Fig. 1(c) is that the scattering state is excited on one of the nucleons by an exchanged meson, and the scattering meson is transformed into the final-state photon via the photoproduction T matrix. If the intermediate meson is in the resonant $\frac{3}{2}$-$\frac{3}{2}$ scattering state, then the orbital angular momentum of the center of mass of the scattering state with respect to the other nucleon is uniquely two (which means that the two nucleons are predominantly in a relative $^3D_1$ state). The photoproduction process, in the magnetic dipole approximation, can then proceed only through the d state of the deuteron. The process corresponding to Fig. 1(b) cannot conserve isotopic spin if only the $\frac{3}{2}$-$\frac{3}{2}$ resonant scattering state is included and will not be considered further.

Because there is no dependence on the nucleon momentum other than the momentum conservation delta functions, this contribution to the transition amplitude is easily written in the relative coordinate representation,

$$-i(\mu_p-\mu_n)\left(\frac{\omega_\gamma}{\gamma}\right)^{1/2}\int d^3x\psi_D^*(x)I\psi(x)\,,\quad(23)$$

where

$$I=\int d^3ld^3m\frac{\exp(i\mathbf{l}\cdot\mathbf{x})\boldsymbol{\sigma}^2\cdot\mathbf{l}\tau_l^2v(l)T_{-\gamma\times\epsilon}^{*1}(m)T_l^1(m)}{(2\omega_l)^{1/2}\omega_l(\omega_l+\omega_m)}\,.$$

The superscripts on the spin and isotopic spin operators indicate which nucleon wave function they are to operate on. The "isobar" approximation will be used, which means replacing the ω_m in the denominator by ω_0, the energy of the resonance. The angular integrals can be done without difficulty, and lead to the expression

$$i(\mu_p-\mu_n)\boldsymbol{\tau}_3{}^1\boldsymbol{\sigma}^1\cdot\boldsymbol{\gamma}\times\boldsymbol{\varepsilon}\alpha_3f^2(9\pi\sqrt{2})^{-1}\times\int_0^\infty drw(r)F(r)U(r)\,,\quad(24)$$

where $w(r)$ is the D-wave deuteron radial wave function, and $U(r)$ is the $^1S_0$ initial-state radial wave function. The function $F(r)$ is defined by

$$F=\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\int\frac{d^3l}{2\pi^2}\exp(i\mathbf{l}\cdot\mathbf{x})\frac{v^2(l)}{\omega_l^2(\omega_l+\omega_0)}\,,$$

[11] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570, 1579 (1956); and G. C. Wick, Rev. Mod. Phys. **27**, 339 (1955).

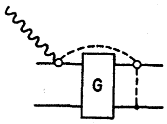

FIG. 2. The contribution of the excited Heitler-London state to the capture amplitude. The box represents the propagation of the excited state through the nuclear potential.

and α_3 is defined by the integral[12]

$$f^2\alpha_3=12\int_0^\infty \frac{dm\,\sin^2\delta_3(m)}{m^2\omega_m v^2(m)}\,.$$

IV. THE ROLE OF THE EXCITED STATE

The excited state contributes to the capture amplitude a term

$$\Sigma_k \psi_D{}^\dagger \langle\Phi|J_\gamma|\Phi_k\rangle \varphi_k\,, \tag{25}$$

where the normalization factors Γ have been set equal to one, and the excited-state amplitude is given formally by

$$\varphi_k=-(K+U+\omega_k-E)^{-1}W_k\psi\,. \tag{16}$$

The problem breaks into three separate parts: the excitation of the excited state, described by $W_k\psi$; the propagation of the excited state, described by the Green's function $(K+U+\omega_k-E)^{-1}$; and the de-excitation of the excited state by the interaction with the photon, given by $\langle\Phi|J_\gamma|\Phi_k\rangle$. A diagram is shown in Fig. 2. If the excitation is calculated up to one-meson exchange, W_k is given by

$$\Sigma_l\langle p_2k|V_l|p_2'\rangle\langle p_1|V_l{}^\dagger|p_1'\rangle_{as}[\omega_l{}^{-1}+(\omega_l+\omega_k)^{-1}]\,. \tag{26}$$

The subscript *as* here means to add the terms with nucelon variables interchanged, but the meson variable k is not interchanged. By the same angular-momentum argument as used to describe the scattering intermediate state in the previous section, it is found that the meson in the excited Heitler-London state can be in the $\frac{3}{2}$-$\frac{3}{2}$ resonant state with respect to one of the nucleons. Only this state will be calculated, and again the isobar approximation will be made whereby the ω_k in the denominators of Eqs. (16) and (26) is replaced by ω_0, the energy of the resonance. In the relative nucleon coordinate representation, the excitation of the excited-state amplitude is

$$W_k\psi=\int\frac{d^3l}{(2\pi)^3}\exp(-i\mathbf{l}\cdot\mathbf{x}+i\mathbf{k}\cdot\tfrac{1}{2}\mathbf{x})\left[\frac{1}{\omega_l}+\frac{1}{\omega_l+\omega_0}\right]$$
$$\times[V^1(l)T_l{}^2(k)+V^2(-l)T_{-l}{}^1(-k)]\,. \tag{27}$$

The angular momenta of the particles in this intermediate state are very complicated and it is convenient to reject all but the dominant partial waves. Apart from the meson phase factor, $\exp(i\mathbf{k}\cdot\frac{1}{2}\mathbf{x})$, the nucleons are in a ${}^3D_1$, $T=1$ state and the meson is in a p-wave state with respect to the center of mass of the two nucleons. The meson phase is expanded into partial waves, and only the s and d waves kept. (The p wave is of opposite parity and incoherent with the principal term.) From this the meson p wave with respect to the center of mass of the nucleons is kept, the nucleons are in a state which is a mixture of ${}^3S_1$ and ${}^3D_1$. In this approximation, the excitation is given by

$$W_k\psi=f(k)H(r)[j_2+(j_0-j_2)S_{12}/4]\mathbf{k}\cdot(\boldsymbol{\sigma}^1-\boldsymbol{\sigma}^2)\psi\,, \tag{28}$$

where

$$f(k)=\frac{2if\exp(i\delta_3(k))\sin\delta_3(k)}{3k^3v(k)(2\omega_k)^{1/2}}\,,$$

$$H(r)=\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\int\frac{d^3l}{2\pi^2}\,\frac{\exp(i\mathbf{l}\cdot\mathbf{x})}{\omega_l}\left(\frac{1}{\omega_l}+\frac{1}{\omega_l+\omega_0}\right),$$

S_{12} is the usual tensor operator, and j_0 and j_2 are spherical Bessel functions of argument $\frac{1}{2}kr$.

In the relative coordinate representation, and where the Green's function is a two-by-two matrix in the angular-momentum variable, it satisfies the radial equation

$$\left(\frac{-d^2}{mdr^2}+\frac{l(l+1)}{r^2}+U(r)+\omega_0-E\right)G(r,r')=\delta(r-r')\,. \tag{29}$$

$U(r)$ is the nuclear potential for the triplet-even states and is characterized by a large tensor potential, it is nondiagonal in the angular momentum and gives the coupling between ${}^3S$ and ${}^3D$ waves. An approximate solution for the Green's function is given in the next section, for the moment we will assume it known.

The de-excitation of the excited state, to lowest order in meson exchanges, is given by

$$\langle\Phi|J_\gamma|\Phi_k\rangle=\tfrac{1}{2}\langle p_2'|p_2\rangle\langle p_1'|J_\gamma|p_1k\rangle_{as}\,,$$

where the magnetic dipole photoproduction matrix element will be approximated by Eq. (22). In the relative coordinate representation, the contribution to the capture amplitude is

$$2(2\pi)^{-3}\int d^3kd^3x\,\exp(i\mathbf{k}\cdot\tfrac{1}{2}\mathbf{x})\psi_D{}^\dagger(x)J_\gamma{}^1(k)\psi_k(x)\,, \tag{30}$$

where

$$J_\gamma{}^1(k)=(\mu_p-\mu_n)(2fv(\gamma))^{-1}(\omega_\gamma/\gamma)^{1/2}T_{-\gamma\times\epsilon}{}^{1*}(k)\,.$$

If the phase factor $\exp(i\mathbf{k}\cdot\frac{1}{2}\mathbf{x})$ is calculated at resonance energy and expanded into partial waves, only the s and d waves being kept, then the angular integrations in

[12] M. Cini and S. Fubini, Nuovo Cimento 3, 764 (1956).

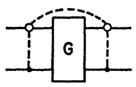

FIG. 3. The contribution of the excited Heitler-London state to the nuclear potential. The box indicates that the nucleons move through the nuclear potential during the lifetime of the excited state.

Eq. (30) may be done, resulting in

$$C\int_0^\infty dr(U_D,w)\begin{pmatrix}0 & 0\\ \sqrt{2}j_2 & j_0-j_2\end{pmatrix}\times\int_0^\infty dr' G(r,r')\begin{pmatrix}\sqrt{2}j_2\\ j_0-j_2\end{pmatrix}H(r')U(r'),\quad(31)$$

where

$$C=2i\alpha_3 f^2(\mu_p-\mu_n)\boldsymbol{\tau}_3{}^1\boldsymbol{\sigma}^1\cdot\boldsymbol{\gamma}\times\boldsymbol{\varepsilon}(729\pi^2\gamma)^{-1/2}.$$

Notice that again only the d state of the deuteron contributes. The sum of (31) and (24) give the interaction effect, in the present approximations exclusively due to the $\frac{3}{2}$-$\frac{3}{2}$ resonant intermediate state.

It was remarked earlier that the recoupling of the excited state to the basic state contributes to the nuclear potential a term

$$V'=-\Sigma_k W_k{}^\dagger G W_k.$$

Since we have already calculated G and W_k, we can easily calculate this potential also. In our approximations we keep only the resonant $\frac{3}{2}$-$\frac{3}{2}$ pi-nucleon intermediate state and this potential acts only in the $T=1$ singlet even two-nucleon states. The potential corresponds to the diagram of Fig. 3. Using the form (28) for the function W_k, the angular integrations are easily done resulting in

$$V'=\frac{2\alpha_3 f^4}{27\pi^2}\frac{H(r)}{r}(\sqrt{2}j_2,j_0-j_2)G(r,r')\begin{pmatrix}2j_2\\ j_0-j_2\end{pmatrix}\frac{H(r')}{r'}.\quad(32)$$

This is of course nonlocal and energy-dependent due to the Green's function.

V. NUMERICAL RESULTS AND CONCLUSIONS

The Green's function of the previous section is nondiagonal in orbital angular momentum as well as in the radial coordinate. The Green's function can be approximately diagonalized (in the angular variables) by a transformation that diagonalizes the sum of the tensor and centrifugal potentials. A wave function in this representation is

$$\bar\psi=S\psi=\frac{1}{r}\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}u\\ -\\ w\end{pmatrix},$$

where u and w are, respectively, the s and d radial wave functions, and the angle α is a function of the radial coordinate. In this representation, the radial Green's function satisfies

$$\left(\frac{-d^3}{mdr^2}+\begin{pmatrix}V_a & 0\\ 0 & V_b\end{pmatrix}+\omega_0-E\right)\bar G(r,r')=\delta(r-r'),\quad(33)$$

where some small nondiagonal terms have been dropped. This representation is fully discussed by Cutkosky,[10] where it is shown that the deuteron is essentially an eigenstate of the "a" potential and that the "b" potential is dominated by the centrifugal barrier. If $Y(r)$ is the eigenstate of the "a" potential for the energy of the deuteron, then the s and d radial wave functions are given by

$$u(r)=\cos\alpha(r)Y(r),\quad w(r)=\sin\alpha(r)Y(r).$$

In Eq. (33) the wave number, $k=i(m(E-\omega_0-V))^{1/2}$ is large and imaginary, particularly for the repulsive V_b. If V_a is not too attractive, both potentials satisfy the condition

$$(\partial V/\partial r)/k^3\ll 1,$$

and a WKB approximation is possible. This approximation gives the Green's function

$$\tfrac{1}{2}mk^{-1/2}(r)k^{-1/2}(r')\exp\left(-\int_c^r kdx\right)\times\left[\exp\left(\int_c^{r'}kdx\right)-\exp\left(-\int_c^{r'}kdx\right)\right];\quad r>r',\quad(34)$$

where G is symmetric in r and r', and a hard-core boundary condition is imposed at $r=c$. A "local" approximation to this Green's function is given by

$$\delta(r-r')\frac{1-(1+2\bar K)\exp(-2\bar K)}{V-E+\omega_0},\quad(35)$$

where $\bar K=\int_c^r kdr$.

The parameters of the pi-nucleon system are taken to fit the low-energy data; we take[10]

$$f^2/4\pi=0.08,\quad \omega_0=2.1,$$
$$\alpha_3=2.6,\quad V(l)=(\lambda^2-1)/(\lambda^2+l^2),\quad \lambda=7.$$

The nucleon-nucleon parameters needed are the incident wave function, the deuteron wave function, and the nuclear potentials V_a and V_b. The incident wave function is taken from Austern and Rost,[1] who integrated the Schrödinger equation using the Gamel-Thaler potential. It is

$$U(r)=1+0.04113r-1.7194\exp(-2.43r)\quad r>c=0.283$$
$$=0\quad r<c.$$

$Y(r)$, the solution of the "a" Schrödinger equation is chosen as a modified Hulthen wave function,

$$Y(r)=N(\exp[-\alpha(r-c)]-\exp[-\beta(r-c)])\quad r>c$$
$$=0\quad r<c.$$

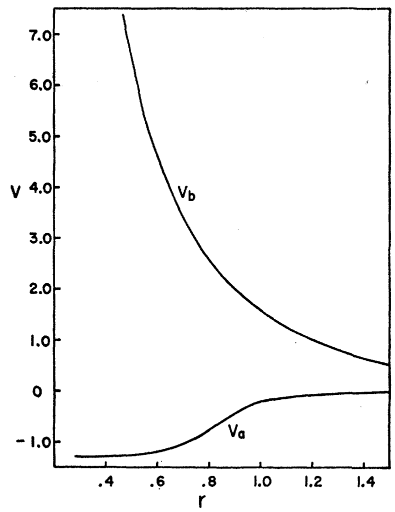

FIG. 4. The potentials V_a and V_b in units of μc^2 versus r in units of $\lambda_\pi = 1.4 \times 10^{-13}$ cm.

α is chosen to give the correct binding energy and β to give the correct effective range. V_a is taken to fit the one-pion-exchange potential for $r>1$, and for $r<1$ is adjusted to both give the correct deuteron binding energy and not be so singular as to invalidate the WKB approximation. V_b is taken from Ref. 10, in the inner region it is dominated by the centrifugal potential, in the outer region the perturbation calculation is expected to be valid. These potentials are shown in Fig. 4. $\alpha(r)$ is taken from Young and Cutkosky.[5] In the outer region it matches the D state of Iwadare *et al.*,[13] in the inner region it is consistent with meson-theoretical calculations of the tensor potential, but adjusted to give several D-state probabilities.

The WKB approximation, Eq. (34), is used for the "a" Green's function. Since V_b is so large and repulsive the local approximation, Eq. (35), is used for the "b" Green's function. The integrands of the two contributions to the capture amplitude are shown in Fig. 5, the importance of the inner regions of the nucleons is evident. Table I gives the interaction current amplitudes.

TABLE I. The interaction effect expressed as percent of the classical amplitude for several values of P_D.

P_D	Contribution of the basic H-L state, Eq. (23)	Contribution of the excited H-L state, Eq. (31)	Total
5.9%		1.05	
6.4%	0.25	1.20	1.45
6.9%		1.70	

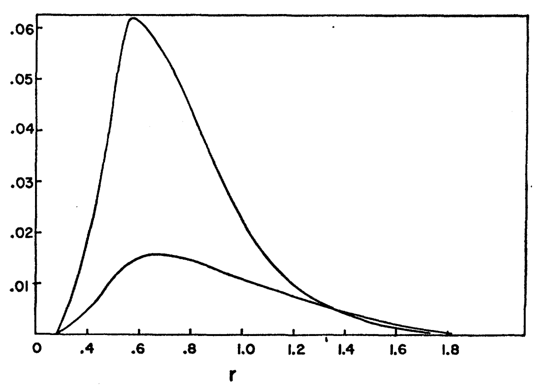

FIG. 5. The integrands of the two contributions to the interaction current amplitude. The larger area is from the excited H-L state, the smaller area from the basic H-L state.

The local Green's function is used to calculate the contribution of the excited state to the nuclear potential. This potential is shown in Fig. 6, where it is compared to that calculated[10] using a Green's function $(\omega_0)^{-1}$. A velocity-dependent correction to the local approximation nowhere exceeds 4% of the reduced mass and indicates that the local approximation shows the most important features of this potential. The contribution of the basic Heitler-London to the $^1S_0$ potential is also shown. The sum of these, besides having the correct long-range behavior, seem compatible with the $^1S_0$ wave function chosen.

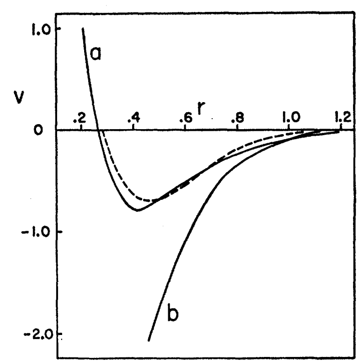

FIG. 6. Contributions to the singlet even potential. (a) is that of the basic H-L state. (b) Contribution of the excited H-L state using $G=\omega_0^{-1}$. (c) The dashed line is the contribution of excited H-L state using the local WKB Green's function.

For a $P_D=6.4\%$ and a hard core at $c=0.283$, we obtain a total interaction of 1.45% to the amplitude, or 2.9% to the cross section. The interpretation of this result must be based on an understanding of the model used for the two-nucleon system. Only the pi-meson interaction was considered, and this was treated with the static-source theory. This nonrelativistic approach does not treat nucleon recoil effects systematically, and leaves out relativistic effects of the motion of the nucleons and nucleon antinucleon pairs. Further, only the $\frac{3}{2}$-$\frac{3}{2}$ resonant scattering state was used, and only one-meson exchange terms were calculated. The model should be valid for low-energy problems and for effects involving the outer regions of the nucleon structure, when the nucleons are not much closer than about one-pion Compton wavelength.

The relativistic corrections to the nucleon currents

[13] J. Iwadare, S. Otsuki, R. Tamagaki, and W. Watari, Progr. Theoret. Phys. (Kyoto) **32**, Suppl. No. 3 (1956).

were calculated by Sugawara[3] and were small, -0.5% of the amplitude. The inner region of the nucleon structure has not been completely neglected, because the classical currents of the whole nucleon have been phenomenologically included in the classical term of the amplitude.

The two-meson exchange interaction currents were not calculated for several reasons. They are too difficult, and the calculation would be incomplete until more is known about the meson-meson interaction. The calculation of a similar matrix element,[5] the magnetic moment of the deuteron, using this same model, gave the result that the two-meson exchange effects were small. This is encouraging to the hope that they would not be important in our case.

This calculation gives an account of that part of the interaction effect due to the $\frac{3}{2}$-$\frac{3}{2}$ scattering state. The result is about one third of that predicted by Austern and Rost,[1] about one-half of that suggested by Partovi,[14] and about twice that calculated by Sugawara.[3] The excited Heitler-London state gives the dominant effect, the principal uncertainty of this contribution is in the source function, the strength with which the excited state is produced. Even in the one-meson exchange approximation this is large only for small internucleon separations, it is quite possible that other short-range interactions could increase the source function. In such a way the result could perhaps be doubled, with a consequent deepening of the potential.

The general conclusion is that the single-meson excited Heitler-London state is important in $T=1$ states. This is particularly seen in the calculated potential, but not so much in the capture amplitude. While the present calculation does not seem to give a sufficiently large interaction effect, it does not rule out the possibility that with more information on the inner structure of nucleons an effect twice as great could be found.

[14] F. Partovi, Ann. Phys. (N. Y.) **17**, 79 (1964).

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Unitarity Bounds of the Scattering Amplitude and the Diffraction Peak

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From unitarity alone a lower bound for the derivative of the absorptive part of the forward scattering amplitude with respect to the momentum transfer is obtained, in terms of the elastic and total cross sections. Comparison with high-energy scattering experiments shows that the actual value of this derivative is rather close to the lower bound, which provides some information on the partial-wave distribution. Our result can also be used to obtain consistency requirements on theoretical models. If Regge behavior is assumed for high-energy scattering, namely, $F(s,t)\simeq f(t)s^{\alpha(t)}$, then one can show that either $\alpha'(0)\geqslant\epsilon>0$ or $\alpha(t)\equiv\text{const}$.

I. INTRODUCTION

TWO qualitative features of high-energy scattering have been known for some time: (i) At a given energy the total cross section and the width of the diffraction peak may not assume arbitrary values. The larger the total cross section the greater is the minimum number of partial waves required to build it up, which means a larger "radius" of the scattering object and consequently a narrower diffraction peak. An expression of such a relationship in the form of an inequality was given in a previous paper.[1,2] (ii) For a given total cross section the width of the diffraction peak increases as one increases the total elastic cross section.[1,2]

A rough estimate of the width Δ of the diffraction peak is indeed easily obtained from:

$$\sigma_{\text{el.}}=\frac{2\pi}{s}\int_{-4k^2}^{0}|f(s,t)|^2\frac{dt}{2k^2}\approx\frac{2\pi}{s}|f(s,0)|^2\frac{\Delta}{2k^2}\geqslant\frac{\Delta}{4}\frac{\sigma_{\text{tot}}^2}{4\pi},\quad(1)$$

which gives

$$\frac{1}{\Delta}\approx\frac{1}{4}\frac{\sigma_{\text{tot}}}{4\pi}\left(\frac{\sigma_{\text{tot}}}{\sigma_{\text{el.}}}\right).\quad(2)$$

In Sec. II, we give a precise meaning to such a rela-

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[1] A. Martin, Phys. Rev. **129**, 1432 (1963).
[2] E. Leader, Phys. Letters **5**, 75 (1963).